TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 3

MATHIAS BRAUN AND WENHAO ZHAO

Homework 3.1 (Construction in the proof of Mittag-Leffler's theorem*)**.** Construct explicitly (in the sense of an explicit series) a holomorphic function $f\colon \mathbf{C}\setminus\{\sqrt{n} : n\in\mathbf{N}\}\to\mathbf{C}$ such that for every $n\in\mathbf{N}$, the function f has the principal part

$$q_n(z) := \frac{\sqrt{n}}{z-\sqrt{n}}$$

at $z_n := \sqrt{n}$[1].

Homework 3.2 (Partial fraction decomposition of $\pi^2/\sin^2(\pi z)$)**.** The goal of this exercise is to prove the formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2}.$$

This will be achieved in several steps. Keep in mind the formula

$$\sin(z) = \frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right].$$

a. Show the assignment $f(z) := \pi^2/\sin^2(\pi z)$ has its singularities exactly in $\mathbf{Z}$ and determine the principle parts of the Laurent series expansion in those points.
b. Show the series

$$g(z) := \sum_{n\in\mathbf{Z}} \frac{1}{(z-n)^2}$$

converges locally uniformly on $\mathbf{C}\setminus\mathbf{Z}$, so that it is meromorphic. Conclude the difference $g(z) - f(z)$ can be extended to an entire function.
c. Show $f(z)$ converges to zero as $|\Im z|\to\infty$ uniformly in $\Re z$.
d. Show the same statement for the function g. Then prove that the difference $g - f$ is bounded on $\mathbf{C}$ and conclude the proof[2].

Date: October 7, 2024.

[1]**Hint.** Prove that a second order Taylor polynomial p_n of q_n yields the local normal convergence of the sum $f = \sum_{n\in\mathbf{N}}[q_n - p_n]$, cf. Theorem 2.2.

[2]**Hint.** Note that $(g-f)(z+1) = (g-f)(z)$ for every $z\in\mathbf{C}$.

Homework 3.3 (Weierstraß elliptic function)**.** Let $\omega_1, \omega_2 \in \mathbf{C}$ be $\mathbf{R}$-linearly independent. Show that up to an additive constant there exists one and only one holomorphic function[3] $\wp\colon \mathbf{C} \setminus \{m\omega_1 + n\omega_2 : m, n \in \mathbf{Z}\} \to \mathbf{C}$ such that

a. $\wp$ has principal part $q_1(z) := 1/z^2$ in $d_1 := 0$ and
b. $\wp$ is $\{\omega_1, \omega_2\}$-periodic, i.e. for every $z \in \mathbf{C} \setminus \{m\omega_1 + n\omega_2 : m, n \in \mathbf{Z}\}$,

$$\wp(z + \omega_1) = \wp(z),$$
$$\wp(z + \omega_2) = \wp(z).$$

You can use without proof that

$$\sum_{\substack{n,m \in \mathbf{Z} \\ |n|+|m| \neq 0}} \frac{1}{(n^2 + m^2)^{3/2}} < \infty.$$

[3] If we require that the zeroth order coefficient of the Laurent series in the origin vanishes, this function is called the Weierstraß $\wp$-function.